The Higher Dimensional Positive Mass Theorem I

by

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1. Introduction

The aim of this paper is to present a proof of the positive mass conjecture in every dimension and without additional topological assumptions.

This conjecture appeared in the early 60s when the definition of the total energy by Arnowitt,Deser and Misner [ADM] was generally accepted.
In the late 70s Schoen and Yau [SY1-2], slightly later Witten [W] and more recently Huisken and Ilmanen [HI] were able to prove the 3 dimensional (Riemannian) positive mass conjecture by conceptually different methods. Its meaning is (by some reduction technique) that an isolated 4 dimensional gravitational system has non-negative total energy. Also in this dimension there are interesting variants of proofs cf.([HI],p.355-356) for a brief overview.

However extensions of the original form of general relativity starting from Kaluza-Klein theory for electrodynamics up to string theories already need 5, 10, 11 or even 26 space-time dimensions to work with and thus the result in higher dimensions is still of significant interest in physics.

In differential geometry it is of equal importance in arbitrary dimensions anyway. It is a key element in scalar curvature geometry as it precisely says that locally one *cannot* increase scalar curvature even if one allows topological modifications (cf. [L1], [L2] for a discussion): geometrically this result means that a complete manifold of non-negative scalar curvature, $Scal \geq 0$, becoming Euclidean at infinity is just the flat $\mathbb{R}^n$.

It was possible to extend the Schoen-Yau proof via minimal hypersurface techniques up to dimension 8 [SY3], [Sm]. But from that on ever-increasing problems due to uncontrollable minimal hypersurface singularities made this

approach intractable. On the other hand, Witten's approach, while working in every dimension [PT], relies on the existence of a spin structure which is not a problem in dimension 3 (every orientable 3-manifold is spin) but becomes an serious restriction in higher dimensions. Finally, the inverse mean curvature flow of Huisken-Ilmanen presently appears in a sense *too* sensitive (it is comparable to the Ricci flow) for this problem in dimensions ≥ 4.

The strategy in this paper is to build on the minimal hypersurface idea of [SY 1-3] to get a tool that works in arbitrary dimensions to treat the positive mass conjecture. The new ideas are substantially based on our recently developed tools (in [CL]) to handle scalar curvature effects near the singular set and employs Allard regularity as a construction aid providing decompositions of minimal hypersurfaces in regular and singular parts after applying a compactification argument from [L1] which allows us to consider this as a more transparent geometric problem on closed manifolds.

To state it, recall that a Riemannian manifold (M, g) is called *asymptotically flat of order* τ if there exists a decomposition $M = M_0 \cup M_\infty$, with M_0 *compact*, and M_∞ diffeomorphic to $\mathbb{R}^n \setminus B_R(0)$ such that the diffeomorphism provides us with an *asymptotic* coordinate system $\{y^i\}$ on M_∞, i.e. with

$$g_{ij} = \delta_{ij} + O(|y|^{-\tau}),\ \frac{\partial g_{ij}}{\partial x_k} = O(|y|^{-\tau-1}),\ \frac{\partial^2 g_{ij}}{\partial x_k \partial x_l} = O(|y|^{-\tau-2}).$$

There could be several *ends* but for the positive energy theorem there are simple arguments how to reduce to the single end case (cf. [L1]).

If the asymptotically flat end M_∞ has order $\tau > \frac{n-2}{2}$ we can assign an invariant $E(M, g)$, the energy, measuring the asymptotic behaviour of (M, g) near infinity (In the physical context this could be motivated via divergence theorem arguments, to capture the total energy and matches pre-relativistic (Newtonian) calculations for a single point mass cf. [ADM] and [Wa]:

$$E(M, g) = \frac{1}{Vol(S^{n-1})} \cdot \lim_{R\to\infty} \int_{\partial B_R} \sum_{i,j} \left(\frac{\partial g_{ij}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_j} \right) \cdot \nu_j \, dV_{n-1}$$

where $\nu = (\nu_1 \ldots \nu_n)$ is the outer normal vector to ∂B_R the condition $\tau > \frac{n-2}{2}$ intends to model the situation that (asymptotically) the end is free of matter.

It turns out that this (from physical arguments defined (and well-defined cf [B])) energy can be recovered from the asymptotic expansion of the metric:

in asymptotic coordinates we have

$$g_{ij} = (1 + \frac{E_k}{4(n-1)}\rho^{2-n})\,\delta_{ij} + h_{ij},$$

with $\rho = |y|, h_{ij} = 0(\rho^{-n})$, and $\partial_k h_{ij} = 0(\rho^{-n-1})$, and $\partial_k\partial_l h_{ij} = 0(\rho^{-n-2})$.

Thus, geometrically, $E < 0$ resp. $E > 0$ means that M has some hyperbolic resp. parabolic flavor near infinity which will be the basis for the reduction to a purely geometric problem.

Now the positive energy theorem can be stated accurately:

Theorem 1 *Let (M^n, g) asymptotically flat of order $\tau > \frac{n-2}{2}$ with scalar curvature ≥ 0. Then $E \geq 0$ and $E = 0$ if and only if (M, g) is isometric to the Euclidean space $\mathbb{R}^n$.*

The (physical) source why one adds the assumption $Scal \geq 0$ is that (M^n, g) arises from a reduction from a $n+1$-dimensional space-time where (M^n, g) forms a maximal Cauchy hypersurface and one assumes that the universe satisfies the so-called dominant energy condition and this gives (by some elementary curvature identities) that the induced metric g has $Scal \geq 0$.

This theorem can be reduced to a crisp geometric *non-existence* result of $Scal > 0-$islands which is the version we are going to prove later on.

Theorem 2 *There is **no** complete manifold (M^n, g) with $Scal > 0$ in the non-empty interior of some compact set $K \subset M^n$ such that $(M^n \setminus K, g)$ is isometric to $(\mathbb{R}^n \setminus B_r(0), g_{Eucl})$ for some $r > 0$.*

Remark 1 Adding some considerably different techniques to this paper we can show two different types of extensions:
1. A *non-classical* generalized version of these results, namely that the *compactness* assumption in Theorem 2 can be substituted for *completeness* (cf. [L3]) which e.g. includes cases with *infinitely* many ends.
2. For *physical* applications we note that all this corresponds to a space-like *maximal* hypersurface in space-time i.e with mean curvature zero. In this context there is also the more general situation with variable mean curvature which will be subject of another paper (cf. [L4]) which will also cover the positivity of the Bondi mass.

Now we start with some preparations and then we *outline* the main steps in the proof: We assume that for some end $E < 0$ and lead this to a contradiction. We first reduce Theorem 1 to Theorem 2: This uses the mentioned hyperbolic flavor of an $E < 0$ - end to find a bending that flattens out the end to a Euclidean space while gaining some non-trivial amount of $Scal > 0$. This can be spread on the complimentary part M_0 and we reach a geometry whose non-existence is the claim of Theorem 2. The details of this reduction were already carried out in [L1], Ch.6.

Thus we can entirely focus on the proof of Theorem 2: The next step is to truncate this flat end: for large $L_i \gg 1$, $i = 1, .., n$ (abbreviated by $\Upsilon \gg 1$ for $\Upsilon = (L_1, ..L_n)$) the faces of the non-equilateral cube

$$C_\Upsilon = [-L_1, L_1] \times [-L_1^2 \cdot L_2, L_1^2 \cdot L_2] \times ... \times [-L_1^2 \cdot L_2^2 \cdot ...L_n, L_1^2 \cdot L_2^2 \cdot ...L_n] \subset \mathbb{R}^n$$

(for some fixed chart of the end) are contained in the Euclidean portion of the flattened end. We delete the complement of this cube in this end and identify opposite sides to get a torus component T^c_Υ.
Thus, after some rescaling, the new geometry (V_Υ, g_Υ) looks like a connected sum of flat large torus

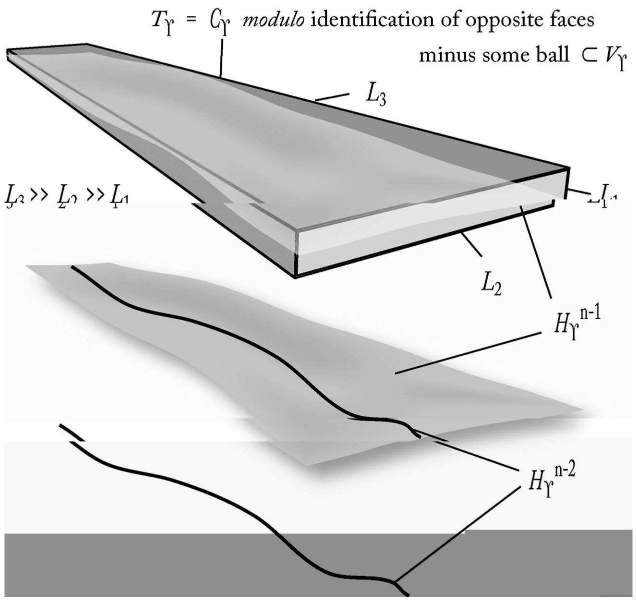
$$T_\Upsilon = (S^1 \times ...S^1, L_1^2 \cdot g_{S^1} \times ... \times L_1^4 \cdot ...L_{n-1}^2 \cdot g_{S^1} \times L_1^4 \cdot L_2^4 \cdot ...L_n^2 \cdot g_{S^1})$$

with a closed manifold W and the metric outside $T^c_\Upsilon = T_\Upsilon \setminus \overline{B_1(p)}$, for some base point p, will have $Scal > 0$. Note that we can increase components of Υ whenever we want without modifying $V_\Upsilon \setminus T^c_\Upsilon =: Q$.
After these preparations we turn to a sketch that for Υ large enough such a V_Υ does actually *not* exist.

The basic tool is the minimal hypersurface technique which allows to gather ambient positive scalar curvature in this lower dimensional object and by induction one iterates this until one reaches a surface (or any other well-understood geometry). In our case we finally get a surface with positive Gauss curvature which contains a torus component, i.e. has genus ≥ 1, which cannot exist.

Of course, here we already reach the problem of those minimal hypersurface singularities which prohibited the use of this method in higher dimensions for decades. Thus we now have to argue in such a way that we never lose touch with the potential singular portions:

For large Υ we can use Allard regularity theory to see that in the homology class the $n-1$-torus $T_q^{n-1} := \{q\} \times (S^1)^{n-1} \subset T_\Upsilon$ with q antipodal to p



there is an area minimizer H^{n-1}_Υ such that $H^{n-1}_\Upsilon \cap T^c_\Upsilon$ is a *smooth* graph (or formally section in the trivial S^1-bundle) over $\{p\} \times (S^1)^{n-1} \setminus \overline{B_1(p)}$ which is also almost isometric to $\{p\} \times (S^1)^{n-1} \setminus \overline{B_1(p)}$.

Thus we indirectly compressed the (potential) singular set $\Sigma \neq \emptyset$ of H^{n-1}_Υ in $H^{n-1}_\Upsilon \cap Q$ (actually a small cylinder containing Q). This compact set Σ has Hausdorff-dimension $\leq n-8$ but there is no structure theory and it is e.g. possible that Σ is a fractal set.

The first step to treat this singular region of H^{n-1}_Υ is a conformal deformation $u_0^{4/n-2} \cdot g_{H^{n-1}_\Upsilon}$ of $H^{n-1}_\Upsilon \setminus \Sigma$ by an eigenfunction > 0 of a generalization of the conformal Laplacian adapted to the singular case:

$$-\triangle u_0 + \frac{n-3}{4(n-2)} Scal_{H^{n-1}_\Upsilon} u_0 = \lambda_0 \cdot |A|^2 \cdot u_0$$

for a certain $\lambda_0 > 0$, where $|A|^2 = \Sigma_{i=1}^{n-1} a_i^2$, $a_i = i\text{-}th$ principal curvature of the second fundamental form A of H^{n-1}_Υ.

It turns out that this deformation nearly keeps the geometry on $H^{n-1}_\Upsilon \cap T^c_\Upsilon$ and this provides us with the geometric condition for an induction argument. However, more significantly, since this is the place where we bypass the old problem of singularities, we can now apply the mechanisms of [CL] to handle

the singular part:
Subsequent to the deformation $u_0^{4/n-2} \cdot g_{H^{n-1}_\Upsilon}$ of $H^{n-1}_\Upsilon \setminus \Sigma$ we deform H^{n-1}_Υ close to Σ by local arguments (which use more advanced geometric measure theory) in a style that can geometrically be described as a stratified version of *codim* ≥ 3 surgery for positive scalar curvature along an (actually augmented) singular set. In the classical regular case one gets a totally geodesic boundary keeping the scalar curvature > 0. The counterpart we obtain is an implicit barrier for $(n-2)$-dimensional minimal hypersurfaces $\subset H^{n-1}_\Upsilon$ homologically equivalent to the boundary of a neighborhood of Σ^{n-7}.
From this one gets a *smooth* $(n-2)$-dimensional hypersurface $N^{n-2} \subset H^{n-1}_\Upsilon$ with positive mean curvature homologically equivalent to that boundary and arbitrarily close to Σ^{n-7}. Finally one transforms a small one sided tube of N^{n-2} into a totally geodesic border (and additionally gives some extra $Scal(g) > 0$). Gluing this with a mirrored copy we get a smooth closed manifold with $Scal > 0$ that contains a (obviously two) nearly flat torus components.
Finally, the non-equilateral largeness assumption on the torus allows us to repeat the argument inductively.

For notational convenience we assume that our manifolds are *oriented*. This is not a restriction since the ends are mapped isometrically under the orientation covering.

2. Induction via Allard's regularity theorem

Here we want to show that the non-equilateral torus component of $V_\Upsilon = T_\Upsilon \# W$ squeezes an area minimizer in such a way that we can show that it is a *smooth* graph over $T^c_\Upsilon = T_\Upsilon \setminus \overline{B_1(p)}$ almost isometric to its base.
This is based on a feature of Allard regularity that says that minimizers close enough to smooth ones will also be smooth. Thus we get a decomposition of the area minimizer in some $n-1$-dimensional smooth component which (moreover) is a torus (which is enough *largeness*-information for the induction step) and a singular part in a small region which can be handled by a certain generalized surgery procedure described in the section 4.

We creep up on the claimed regularity starting from the theory of currents (and varifolds) which also provides a brief look behind the scenes.

First we recall the needed notations (cf. [S]): One defines the space of m-currents $\mathcal{D}_m$ as the dual space of the space of smooth (compactly supported) m-forms $\mathcal{D}^m$ on $\mathbb{R}^n$. To carry this over to manifolds one restricts these definition to open sets U e.g. the elements of $\mathcal{D}^m(U)$ are supported in U.

For a current T on U we define its *support* $spt\, T := U \setminus \cup W$, where the union $\cup W$ is taken over all open sets W with $T(w) = 0$ whenever $w \in \mathcal{D}^n(U)$ with *supp* $w \subset W$. The (weighted) area of a current, its mass $\mathbf{M}_U(T)$, will be defined by $\mathbf{M}_U(T) = \sup_{|w|\leq 1, \mathrm{supp} w \subset U} T(W)$ (cf. [S], 26.6, for some remarks concerning this definition).

We single out a subcategory of more geometric objects which also satisfy compactness results for the flat norm (see below): $T \in \mathcal{D}_m(U)$ is called an *integral current* (all of them form the set $\mathcal{I}_m(U)$) if there is a $m-$rectifiable subset N of U (whose (suitably defined) boundary is also rectifiable) and a measurable function $\xi : N \to \Lambda_m(\mathbb{R}^n)$, which in almost every point of N forms a normalized volume form of N such that for any $\omega \in \mathcal{D}^m(U)$:

$$T(\omega) = \int_N < \omega(x), \xi(x) > d\mathcal{H}^m$$

The basic existence (which is a consequence of the compactness result) and regularity theorem of *codim 1* area minimizers in a compact manifold M^n (= minimizers of $\mathbf{M}_M(T)$, $T \in \mathcal{I}_m(M)$) is (cf. [F] CH.5, [G])

Proposition (2.1) *For any $\alpha \in H_{n-1}(M^n, \mathbb{Z})$ there is an area minimizing current $X^{n-1} \in \alpha$ whose support is a smooth hypersurface except for some singular set Σ of codim 8 in M^n.*

(The existence statement refers to a homology defined for currents (cf. [F], Ch.4) which has some geometric extra features like certain types of compactness of homology classes and a quantitative sense of being homologous via the flat norm below. It is only with hindsight that one can interpret X^{n-1} as a cycle representing α in singular homology.)

Now we consider more concretely $V_\Upsilon = T_\Upsilon \# W$. Using a slight deformation we can make sure that X has to hit Q: locally we are in $\mathbb{R}^n$ and we may consider the unit ball B (instead of Q):

Lemma (2.2) *If we have a metric with Scal > 0 near ∂B in $B, g_0 \equiv g_{Eucl}$ outside of B we can deform the metric in any ε-neighborhood $U_\varepsilon(\partial B) \subset B$ of ∂B in such that the metric is deformed to $g_\varepsilon = e^{2f} \cdot g_{Eucl}$ with $f < 0$ on*

$U_\varepsilon(\partial B), g_\varepsilon \equiv g_{Eucl}$ *on* $\mathbb{R}^n \setminus U_\varepsilon(\partial B)$ *and* $Scal > 0$ *on* $U_\varepsilon(\partial B)$.

Proof For large d (which depends on the geometry in B) we can use

$$f_d(x) = \begin{cases} -\exp(-2d/dist(x, \partial B_{1+\varepsilon})) & \text{on } U_\varepsilon(\partial B) \\ 0 & \text{on } \mathbb{R}^n \setminus B_{1+\varepsilon}(0) \end{cases}$$

Now a look at the corresponding transformation law

$$Scal(e^{2f}_{Eucl}) = e^{-2f} \cdot \big(-2(n-1)\Delta f - (n-1)(n-2)|\nabla f|^2 + Scal(g_{Eucl})\big)$$

shows that for large $d \gg 1$ the right hand side will be negative on any given ε-neighborhood $U_\varepsilon(\partial B) \subset B$ of ∂B since the second order derivative in radial direction (in Δf) is positive and dominates all the other terms. (cf. Lemma (3.1) below) $\square$

Thus we may assume that our metric already has this properties of such a g_ε already for Q (instead of a small neighborhood) and we get for any area minimizer X^{n-1} homologous to

Corollary (2.3) *The area minimizing hypersurface* X^{n-1} *has non-empty intersection with* Q*.*

Proof If $X^{n-1} \cap Q = \emptyset$, then we may translate X^{n-1} in the flat T_Υ until it intersects Q with $\mathbf{M}(X^{n-1} \llcorner Q) > 0$.
Since the metric g_{flat} is substituted for $e^{2f} \cdot g_{flat}$ for some $f < 0$ this means that the $(n-1)-$Hausdorff-measures of all sets are multiplied by $e^{(n-1)\cdot f} < 1$ and therefore we have the strict inequality

$$\mathbf{M}(X^{n-1}_{translated} \llcorner Q)_{e^{2f}\cdot g_{flat}} < \mathbf{M}(X^{n-1}_{translated} \llcorner Q)$$

which contradicts the area minimizing property of X^{n-1}. $\square$

For convenience we can adjust the metric in Q (keeping $Scal > 0$) by slightly decreasing the Ricci curvature in normal direction some interior point of Q to turn X into *the unique* minimizer henceforth called $H^{n-1}_\Upsilon \subset V_\Upsilon$ (this is carried out in detail in [Sm]).

Now we check that $spt\, H^{n-1}_\Upsilon$ is squeezed between two narrow parallel flat tori. There is a similar argument in [SY1] where the assumption $E < 0$ for an asymptotically flat space is used to derive height estimates for solutions of Plateau problems. However whereas it is used there to get convergence

(which was non trivial since these spaces are non-compact) we use it here to show that we do not have any singularity in certain parts of $spt\, H^{n-1}_{\Upsilon}$ (which in turn was not an issue in dimension ≤ 7).

Lemma (2.4) *For any $\Upsilon \gg 1$ the tori $T^{n-1}_q := \{q\} \times ...(S^1)^{n-1} \subset T_{\Upsilon}$ for $q \notin B_1(p)$ will not intersect spt H^{n-1}_{Υ}.*

Proof Since H^{n-1}_{Υ} represents $[\![T^{n-1}_q]\!]$ we can lift the embedding of H^{n-1}_{Υ} into $M^n = T_{\Upsilon}\#W$ to $\mathbb{R}\times T^{n-1}\#W$, where we have taken the infinite covering of the S^1-factor perpendicular to T^{n-1}_q. Now the compactness of *spt* H^{n-1}_{Υ} gives a point $y \in spt\, H^{n-1}_{\Upsilon}$ with maximal first coordinate. Then we can take a flat T^{n-1}_q in the flat part of T_{Υ} that runs through this point. But a minimizer cannot (even locally) lie on one side of a hyperplane (if it does not coincide with the hyperplane). $\square$

Now we use a regularity result by Miranda [M] (cf. [G],9.4) (which was later covered by a quantitative statement from Allard Theory of varifolds [A]): if a kind of generalized normal vector to H does not vary to rapidly then this minimal hypersurface is already smooth. And [M] and [A] provide us with a convenient way to verify this: we just need to show that a minimizer is close enough to a smooth one.

The notion of closeness is that of the flat norm ([S],Ch.31): Let W, U be an open subsets of V_{Υ} with $\overline{W} \subset U$, then we define the pseudometrics (which induce a topology called the *flat norm topology*) by $d_W(C_1, C_2) :=$

$$\inf\{\mathbf{M}_W(S) + \mathbf{M}_W(R) | C_1 - C_2 = S + \partial R, S \in \mathcal{I}_{n-1}(V_{\Upsilon}), R \in \mathcal{I}_n(V_{\Upsilon})\}$$

In order to apply this we use the particular definition of V_{Υ}: Rescale V_{Υ} by L_1^{-4} (i.e. multiply lengths L_1^{-2}). This scaling transforms T_{Υ} into a very thin (in the first coordinate) torus

$$(S^1 \times ...S^1, L_1^{-2} \cdot g_{S^1} \times L_2^2 \cdot g_{S^1} \times ... \times L_2^4 \cdot ...L_{n-1}^2 \cdot g_{S^1} \times L_2^4 \cdot L_2^4 \cdot ...L_n^2 \cdot g_{S^1})$$

which for $L_1 \to \infty$ collapses to an $n-1$-dimensional one $T^{n-1}_{L_2,..L_n}$ while the component $Q \subset V_{\Upsilon}$ just shrinks according to this scaling.
But that means that we can identify increasing parts of the $T^{n-1}_{L_2,..L_n} \setminus L_1^{-2} \cdot Q$ considered as submanifolds of V_{Υ} for $L_1 \to \infty$.

Now consider the minimizer $H^{n-1}_{\Upsilon} \subset V_{\Upsilon}$

Lemma (2.5) *There is an* $R_\Upsilon \in \mathcal{I}_n(V_\Upsilon)$ *with* $[\![T^{n-1}_{L_2,..L_n}]\!] - [\![H^{n-1}_\Upsilon]\!] = \partial R^n$ *such that*

$$\mathbf{M}_U(R_\Upsilon) \le Vol_n(V_\Upsilon)$$

Proof We show that (in measure theoretic sense expressed by multiplicities of currents see below) the mapping degree of $\pi : sptH^{n-1}_\Upsilon \setminus l_1^{-2} \cdot B_1^{n-1}(p) \to T^{n-1}_{L_2,..L_n} \setminus l_1^{-2} \cdot B_1^{n-1}(p)$, ($l_1 \gg 1$ and $L_1 \ge l_1$) is ± 1

If we knew that both are smooth manifolds this were trivial. But the *representatives* of the used homology theories (although isomorphic since both satisfy the Eilenberg-Steenrod axioms (cf. [F],Ch.4)) differ in their differential topological behaviour.
Thus one uses other tools: The condition $[\![T^{n-1}_{L_2,..L_n}]\!] - [\![H^{n-1}_\Upsilon]\!] = \partial R^n$, with R^n n-rectifiable (which says R^n is a union of open subsets modulo some set of measure zero) implies that π is surjective (otherwise points in $T^{n-1}_{L_2,..L_n}$ become interior and boundary points of the set $sptR^n$).
Since $T^{n-1}_{L_2,..L_n} \setminus l_1^{-2} \cdot B_1^{n-1}(p)$ is connected the constancy theorem ([S],Ch.26) gives us $\pi_\#(H^{n-1}_\Upsilon \llcorner T^{n-1}_{L_2,..L_n} \setminus l_1^{-2} \cdot B_1^{n-1}(p)) = \lambda \cdot [\![T^{n-1}_{L_2,..L_n} \setminus l_1^{-2} \cdot B_1^{n-1}(p)]\!]$ with λ an *integer* since we are working with integral currents.

Now for large Υ and $L_1 \gg l_1$ we note that $T^{n-1}_{L_2,..L_n} \setminus l_1^{-2} \cdot B_1^{n-1}(p)$ contains most of the volume of $T^{n-1}_{L_2,..L_n}$, thus a comparison argument using that for Lipschitz maps $f : \mathbb{R}^\mathbf{m} \to \mathbb{R}^\mathbf{n}$ and a Borel subset $A \subset \mathbb{R}^m$ (cf. [F], (2.10.11))

$$\int_{\mathbb{R}^n} \#\{x \in A : f(x) = y\}\, d\mathcal{H}^k y \le Lip(f)^k \cdot \mathcal{H}^k(A).$$

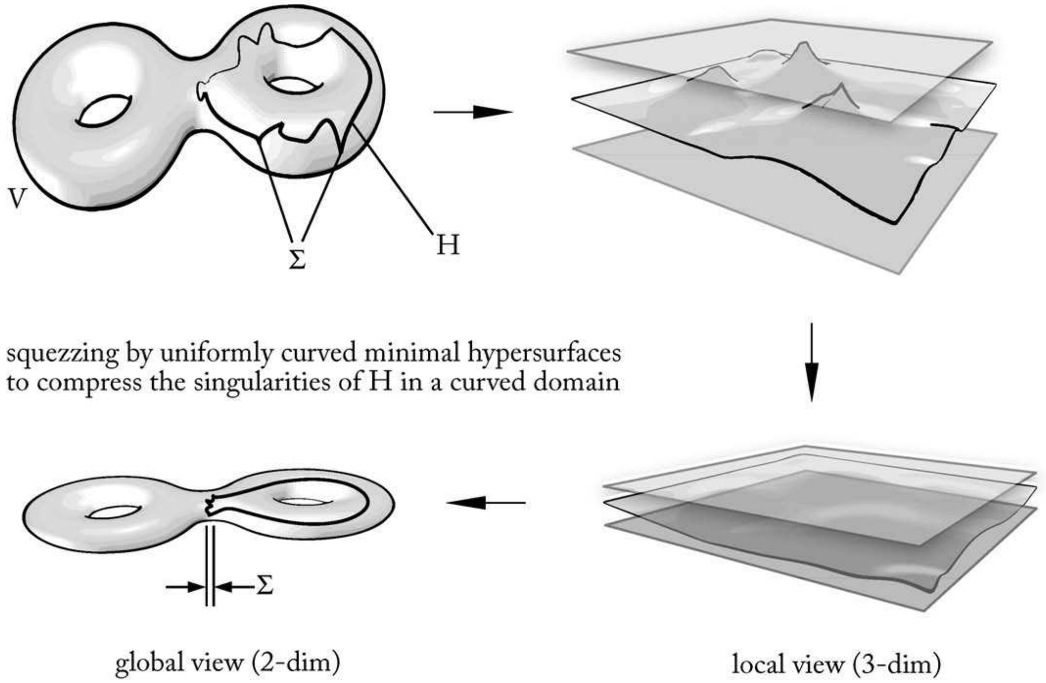
(for any k) shows that $|\lambda| \ge 2$ would imply that $\mathbf{M}(T^{n-1}_{L_2,..L_n}) < \mathbf{M}(H^{n-1}_\Upsilon)$.
If $\lambda = 0$ the subjectivity gives us that for $\Upsilon \to \infty$, $\mathbf{M}(H^{n-1}_\Upsilon) \to \infty$ whereas $\pi_\#(H^{n-1}_\Upsilon \llcorner T^{n-1}_{L_2,..L_n} \setminus l_1^{-2} \cdot B_1^{n-1}(p)) = 0$ gave a universal upper bound for the volume of the minimizer. $\square$

The regularity result for the minimizer $H^{n-1}_\Upsilon \subset V_\Upsilon$ can now be formulated in a version adapted to our problem:

Proposition (2.5) *For any* $\varepsilon > 0$ $l_1 \gg 1$, *there is an* $L_\varepsilon \ge l_1$ *such that for* $L_1 \ge L_\varepsilon$

$$d_{T_\Upsilon \setminus l_1^{-2} \cdot (S^1 \times B_1^{n-1}(p))}(T^{n-1}_{L_2,..L_n} \setminus l_1^{-2} \cdot B_1^{n-1}(p), H^{n-1}_\Upsilon) \le \varepsilon$$

Moreover for any $\delta > 0$ *there is a* $\varepsilon_{l_1,\delta} > 0$ *such that* H^{n-1}_Υ *is a graph of some smooth* F *over* $T^{n-1}_{L_2,..L_n} \setminus l_1^{-2} \cdot B_1^{n-1}(p)$ *for* $L_1 \ge L_\varepsilon$ *with* $|F|_{C^k} \le \delta$.



Proof The first part is immediate from (2.4) and the fact that the volume on the right hand side of $\mathbf{M}_U(R_\Upsilon) \leq Vol_n(V_\Upsilon)$ is scaled by L_1^{-2n}.

Therefore, for the second part, we now know that we may assume that $d_{T_\Upsilon \setminus l_1^{-2} \cdot (S^1 \times B_1^{n-1}(p))}(T^{n-1}_{L_2,..L_n} \setminus l_1^{-2} \cdot B_1^{n-1}(p), H_\Upsilon^{n-1})$ is small enough to apply Allard regularity due to an tilt excess estimate [S],Ch.23,24 which arises from (2.4) and the flat norm convergence to the regular minimal hypersurface $T^{n-1}_{L_2,..L_n} \setminus l_1^{-2} \cdot B_1^{n-1}(p)$ or more transparent from ([G],9.4) where the assumptions of the De Giorgi Lemma are verified (which is the basis of all such regularity results). $\square$

Thus we have that the singular set is contained in the arbitrarily small cylinder $S^1 \times B_1^{n-1}(p)$.

3. Conformal Deformation and Controlled geometry

We start with a slight extension of (2.2) still noting that the second radial

derivative can be made to dominate the total effect:

Lemma (3.1) *Let (M.g) be a manifold with $Scal \geq 0$ and > 0 in some point $p \in M$ then for any open connected neighborhood U of p, any $k \geq 0$ and $\varepsilon > 0$ we can find a conformal deformation $e^{2f} \cdot g$ with $f \equiv 0$ on $M \setminus U$, $\|f\|_{C^k} < \varepsilon$ and $Scal(e^{2f} \cdot g) > 0$ on U.*

Proof Cover U by a series of open pointed sets (V_i, p_i) each diffeomorphic to $(B_1(0), 0)$ with locally finite intersection number such that $p_i \in V_{i+1}$. Now perform on each of these sets a deformation (transplanted from $(B_1(0), 0)$). On $(B_1(0), 0)$ choose for some $\delta_i \ll 1$ some cut-off function $\phi_{\delta_i} \in C^\infty([0,1],[0,1])$, $\phi_{\delta_i} \equiv 0$ near 0 and $\phi_{\delta_i} \equiv 1$ on $[\delta_i, 1]$ and define

$$f_{i,\delta_i}(x) = \begin{cases} -\phi_\delta \cdot \exp(-2d_i/dist(x, \partial B_1)) & \text{on } B_1(0) \\ 0 & \text{on } \mathbb{R}^n \setminus B_1(0) \end{cases}$$

For the first ball (V_1, p_1) assume that the ball $(B_{\delta_1}(0), 0)$ is map into the open set $Scal^{-1}(\mathbb{R}^{>0})$. Then we can choose d_1 that small such that the conformal deformation $e^{2f_{1,\delta_1}}$ keeps $Scal > 0$ in the image of $(B_{\delta_1}(0), 0)$. Outside this deformation increases $Scal$ hence it will be > 0 on the union of previous set $Scal^{-1}(\mathbb{R}^{>0})$ and V_1.

Then, for an inductively chosen pair d_i, δ_i with growing d_i and shrinking δ_i we get the desired deformation $\prod_{i \geq 1} e^{2f_{i,\delta_i}} \cdot g$ by multiplying all the deformations.

The scalar curvature on U for this metric is positive: the relative effect is

$$Scal(\prod_{k \geq i \geq 1} e^{2f_{i,\delta_i}} \cdot g) =$$

$$e^{-2f_{i,\delta_i}} \cdot \left(2(n-1) \cdot \Delta f_{i,\delta_i} - (n-1)(n-2) \cdot |\nabla f_{i,\delta_i}|^2 + Scal(\prod_{k \geq i \geq 1}) e^{2f_{i,\delta_i}} \cdot g \right)$$

For large chosen d_i we find from elementary computations that $\Delta f_{i,\delta_i} > 0$ overcompensates the $|\Delta f_{i,\delta_i}|^2$-term and therefore these two terms yield a *positive* contribution to $Scal$. (cf. [L3], sec.3 for a formal computation)

□

We may apply this for $k = 3$ and small $\varepsilon > 0$ to $(V_\Upsilon, L_1^{-4} \cdot g_\Upsilon)$ to get a metric g^ε_Υ with $Scal_{g^\varepsilon_\Upsilon} > 0$ and $\|L_1^{-4} \cdot g_\Upsilon - g^\varepsilon_\Upsilon\|_{C^k} < \varepsilon$.

Now we realize that we have new minimizer $H^{n-1}_{\Upsilon,\varepsilon}$ for this new geometry. But actually (although there might be substantial changes in Q) if we

now use the quantitative version of Allard regularity ([S], Ch.24) which is not just valid for minimal hypersurfaces but there is a constant $\varepsilon_0(n) > 0$ such that it is also satisfied if for a generalized mean curvature trA one has $\left(\int_B \|trA_H\|^n d\mu\right)^{1/n} \cdot r^{1/n} \leq \varepsilon_0(n)$ for balls B in $T^{n-1}_{L_2,..L_n} \setminus l_1^{-2} \cdot B^{n-1}_1(p)$.

Thus we get that in $T_\Upsilon \setminus l_1^{-2} \cdot (S^1 \times B^{n-1}_1(p))$: using the uniqueness of H^{n-1}_Υ for $\varepsilon \to 0$ the $H^{n-1}_{\Upsilon,\varepsilon}$ will converge in flat norm to the H^{n-1}_Υ and therefore the $H^{n-1}_{\Upsilon,\varepsilon}$ (when considered with respect to g_Υ) are eventually smooth and C^k-near to H^{n-1}_Υ everywhere outside the cylinder $S^1 \times B^{n-1}_1(p)$.

For such a small $\varepsilon > 0$ we want to analyze and modify the scalar curvature on $H^{n-1}_{\Upsilon,\varepsilon}$. Since we have to understand the singular part $\Sigma_{\Upsilon,\varepsilon}$ in Q later on we will introduce a version of the conformal Laplacian which is scaling invariant relative to these singularities.

Since $H^{n-1}_{\Upsilon,\varepsilon}$ is area minimizing the 2nd variation of its area is ≥ 0. Now choose ν is a unit normal vector field and f smooth with support outside $\Sigma_{\Upsilon,\varepsilon}$, then $f \cdot \nu$ is an infinitesimal variation of $H^{n-1}_{\Upsilon,\varepsilon}$. A direct computation gives the expression:

$$0 \leq Area''(f \cdot \nu) = \int_{H^{n-1}_{\Upsilon,\varepsilon}} |\nabla_{H^{n-1}_{\Upsilon,\varepsilon}} f|^2 - f^2(|A|^2 + Ric_{V_\Upsilon}(\nu, v))dA \geq 0$$

$$\iff \qquad \int_{H^{n-1}_{\Upsilon,\varepsilon}} |\nabla f|^2 + \frac{n-3}{4(n-2)} Scal_{H^{n-1}_{\Upsilon,\varepsilon}} f^2 dA \quad \geq$$

$$\int_{H^{n-1}_{\Upsilon,\varepsilon}} \frac{n}{2(n-1)} |\nabla f|^2 + \frac{n-3}{2(n-2)} f^2 \left(|A|^2 + Scal_{V_\Upsilon}\right) dA$$

where $|A|^2 = \Sigma_{i=1}^{n-1} a_i^2$, $a_i = i$-th principal curvature of A.

Thus if $(V_\Upsilon, g^\varepsilon_\Upsilon)$ has $Scal > 0$ we infer

$$\int_{H^{n-1}_{\Upsilon,\varepsilon}} |\nabla f|^2 + \frac{n-3}{4(n-2)} Scal_{H^{n-1}_{\Upsilon,\varepsilon}} \cdot f^2 dA \geq \int_{H^{n-1}_{\Upsilon,\varepsilon}} \frac{n-3}{2(n-2)} f^2 \cdot |A|^2 dA$$

Now we may assume that the metrics and therefore the hypersufaces are analytic, thus we may assume that $|A|^2$ is analytic and its zero set is lower dimensional (otherwise it is totally geodesic and would not have any singularities). Therefore it makes sense to consider the following quotients

$$\lambda_0 := \inf_{f \not\equiv 0, smooth, \operatorname{supp} f \subset H^{n-1}_{\Upsilon,\varepsilon} \setminus \Sigma_{\Upsilon,\varepsilon}} \frac{\int_{H^{n-1}_{\Upsilon,\varepsilon} \setminus \Sigma_{\Upsilon,\varepsilon}} |\nabla f|^2 + \frac{n-3}{4(n-2)} Scal_{H^{n-1}_{\Upsilon,\varepsilon}} f^2}{\int_{H^{n-1}_{\Upsilon,\varepsilon} \setminus \Sigma_{\Upsilon,\varepsilon}} |A|^2 \cdot f^2} \in (1/4, 1)$$

The lower estimate is trivial while the upper bound comes from the identity $|A|^2+2\cdot Ric_{V_\Upsilon}(\nu,\nu) = Scal_{V_\Upsilon} - Scal_{H^{n-1}_{\Upsilon,\varepsilon}}$, one can choose test functions with support in the (sufficiently large chosen) flat part so that one may assume $Ric_{V_\Upsilon}(\nu,\nu), Scal_{V_\Upsilon} = 0$.

Although $H^{n-1}_{\Upsilon,\varepsilon}\backslash\Sigma_{\Upsilon,\varepsilon}$ is an open manifold we can find (actually construct) a smooth function positive (while not necessarily integrable) function $u_{\Upsilon,\varepsilon}$ on $H^{n-1}_{\Upsilon,\varepsilon} \setminus \Sigma_{\Upsilon,\varepsilon}$ with

$$-\triangle u_{\Upsilon,\varepsilon} + \frac{n-3}{4(n-2)} Scal_{H^{n-1}_{\Upsilon,\varepsilon}} u_{\Upsilon,\varepsilon} = \lambda_0 \cdot |A|^2 \cdot u_{\Upsilon,\varepsilon}$$

We observe that $Scal_{u^{4/n-3}_{\Upsilon,\varepsilon}\cdot g_{H^{n-1}_{\Upsilon,\varepsilon}}} > 0$, since:

$$-4(n-2)/(n-3)\cdot \Delta u_{\Upsilon,\varepsilon} + Scal_{g_{H^{n-1}_{\Upsilon,\varepsilon}}} \cdot u_{\Upsilon,\varepsilon} = Scal_{u^{4/n-3}_{\Upsilon,\varepsilon}\cdot g_{H^{n-1}_{\Upsilon,\varepsilon}}} \cdot u^{n+1/n-3}_{\Upsilon,\varepsilon}$$

For the torus component we finally check that we actually already reproduced the start conditions for the lower dimensional induction hypothesis: we will analyze what happened to its geometry under this conformal deformation:

Lemma (3.2) *For any $\eta > 0$ there is an $\varepsilon_\eta > 0$ and $l_1(\eta) \gg 1$, $L_1(\eta) \gg l_1(\eta)$, $L_i(\eta) \gg 1$ such that for $\varepsilon \in (0,\varepsilon_\eta)$ and $L_1 \geq L_1(\eta)$ and $\Upsilon = (L_1, L_2(\eta), ...L_n(\eta))$*

$$\|\Lambda_\Upsilon \cdot u_{\Upsilon,\varepsilon} - 1\|_{C^3(H^{n-1}_{\Upsilon,\varepsilon}\backslash l_1^{-2}\cdot(S^1\times B^{n-1}_1(p)),g^\varepsilon_\Upsilon)} < \eta$$

For some suitable normalizing constants $\Lambda_{\Upsilon,\varepsilon} > 0$.

Proof For $L_1 \to \infty$ we observe that $(H^{n-1}_{\Upsilon,\varepsilon} \setminus l_1^{-2}\cdot(S^1 \times B^{n-1}_1(p)), g^\varepsilon_\Upsilon)$ converges in Hausdorff-topology to the flat $T^{n-1}\setminus\{p\}$ and converges C^k-compactly.
We may assume that $\|u_{\Upsilon,\varepsilon}\|_{L^2(H^{n-1}_{\Upsilon,\varepsilon}\backslash l_1^{-2}\cdot(S^1\times B^{n-1}_1(p)),g^\varepsilon_\Upsilon)} = 1$. In particular $Scal(g_{H^{n-1}_{\Upsilon,\varepsilon}}) \to 0$ and $\Delta u_{\Upsilon,\varepsilon} \to 0$ compact uniformly since $\lambda_0\cdot|A|^2$ also converges to zero.
Thus there is a subsequence that converges to a harmonic function u with $\|u\|_{L^2(H^{n-1}_{\Upsilon,\varepsilon}\backslash l_1^{-2}\cdot(S^1\times B^{n-1}_1(p)),g^\varepsilon_\Upsilon)} = 1$ and by a diagonal sequence argument we may assume that u is defined everywhere on $T^{n-1}\setminus\{p\}$.
But u is bounded: $u > 0$ and the maximum principle says that, for any $\rho > 0$, $\min(u)$ and $\max(u)$ on $T^{n-1}\setminus B_{\rho/2}(p)$ are assumed on $\partial B_{\rho/2}(p)$

and therefore the Harnack inequality applied to the *scale-invariant* situation of ring-regions $B_\rho(p) \setminus B_{\rho/2}(p) \subset B_{2\rho(p)} \setminus B_{\rho/4}(p)$ around p which gives $\max_{B_\rho(p)\setminus B_{\rho/2}(p)} u \leq C \cdot \min_{B_\rho(p)\setminus B_{\rho/2}(p)} u$ for a $C > 0$ independent of ρ. Since $\min(u)$ is upper bounded by the value in some fixed point $y_0 \in T^{n-1} \setminus \{p\}$ this implies $\sup_{T^{n-1}\setminus\{p\}} u \leq C \cdot \inf_{T^{n-1}\setminus\{p\}} u$.
Now the bounded harmonic function u can be extended smoothly giving a harmonic and hence constant function on T^{n-1}. But this implies that $u_{\Upsilon,\varepsilon} \to$ *const.* compactly in C^k on the torus.
Here and above we used the C^k-identification of large parts of $H^{n-1}_{\Upsilon,\varepsilon}$ with those of torus (by projection) which results from Allard regularity. $\square$

4. Doubling and Induction

At this point we have a $Scal > 0$-geometry on $H^{n-1}_{\Upsilon,\varepsilon}$ which looks like a nearly flat regular torus except for a part contained in a little ball B where we may find a non-trivial singular set $\Sigma_{\Upsilon,\varepsilon}$.
The way to handle this part is an analytic extension of the *codim 3* surgery techniques which preserve $Scal > 0$ introduced in [GL] and [SY4] which works for any singular set of *codim* > 2:

In $H^{n-1}_{\Upsilon,\varepsilon} \cap B$ we modify $u_{\Upsilon,\varepsilon}$ in two steps: we first substitute this non-uniquely determined function (note we are on an open manifold) for the smallest possible positive solution of $-\triangle u_{\Upsilon,\varepsilon} + \frac{n-3}{4(n-2)} Scal_{H^{n-1}_{\Upsilon,\varepsilon}} u_{\Upsilon,\varepsilon} = \lambda_0 \cdot |A|^2 \cdot u_{\Upsilon,\varepsilon}$ with the same boundary value along $H^{n-1}_{\Upsilon,\varepsilon} \cap \partial B$. This allows a transition to an infinitesimal level near singular points using tangent cones and get some more insight in the limiting behavior of that function. From this we can start over and deform the metric in a way that the geometry bends open near $\Sigma_{\Upsilon,\varepsilon}$.
In very rough terms this is the path to following doubling result in [CL]:

Proposition (4.1) *For any given $\rho > 0$ there is a neighborhood $V_\rho \subset \varepsilon$-neighborhood of $\Sigma_{\Upsilon,\varepsilon}$ in $H^{n-1}_{\Upsilon,\varepsilon}$ such that the smooth doubling*

$$D_{\rho,\Upsilon,\varepsilon} := H^{n-1}_{\Upsilon,\varepsilon} \setminus V_\rho \cup_\sim H^{n-1}_{\Upsilon,\varepsilon} \setminus V_\rho,$$

($\sim$ means glueing along ∂V_ρ) admits a smooth metric g_ρ with $Scal(g_\rho) > 0$.

For small ρ the metric on either part of $D_{\rho,\Upsilon,\varepsilon}$ deviates from that one obtained in the previous section only in B and actually except for a kind of generalized warped product deformation near ∂V_ρ the metric g_ρ is conformal to the induced metric on $H^{n-1}_{\Upsilon,\varepsilon} \setminus V_\rho$.

Now notice that $(H^{n-1}_{\Upsilon,\varepsilon} \setminus l_1^{-2} \cdot (S^1 \times B_1^{n-1}(p)), u^{4/n-2}_{\Upsilon,\varepsilon} \cdot g_{H^{n-1}_{\Upsilon,\varepsilon}})$ can be assumed as flat (in C^k-topology) as wanted and we have reproduced the start settings but we are now in dimension *(n-1)* and we have two ends. The second end can just be unified with the uncontrolled part Q.

Thus we can enter the induction scheme: the metric on the tori $L_1^2 \cdot g_{S^1} \times ... \times L_1^4 \cdot ... L_{n-1}^2 \cdot g_{S^1} \times L_1^4 \cdot L_2^4 \cdot ... L_n^2 \cdot g_{S^1}$ is reduced in the *i-th* step by deleting the *i-th* factor and scaling by L_i^{-4}. For a suitably chosen collection of lengths $L_1 \ll L_2 .... \ll L_n$ we can repeat the described procedures in each step we finally reach a surface of genus ≥ 1 but with $Scal > 0$ and this is not possible by Gauss-Bonnet which also concludes the proof of Theorem 2. $\square$

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